

Ion dynamics and the magnetorotational instability in weakly-ionized discs

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ABSTRACT

The magnetorotational instability (MRI) of a weakly ionized, differentially rotating, magnetized plasma disc is investigated in the multi-fluid framework. The disc is threaded by a uniform vertical magnetic field and charge is carried by electrons and ions only. The inclusion of ion inertia causes significant modification to the conductivity tensor in a weakly ionized disc. The parallel, Pedersen and Hall component of conductivity tensor becomes time dependent quantities resulting in ac and dc components of the conductivity. The time dependence of the conductivity causes significant modification to the parameter window of magnetorotational instability.

The effect of ambipolar and Hall diffusion on the linear growth of the magnetorotational instability is examined in the presence of time dependent conductivity tensor. We find that the growth rate in the ambipolar regime can become somewhat larger than the rotational frequency, especially when the departure from ideal MHD is significant. Further, the instability operates on large scale lengths. This has important implication for angular momentum transport in the disc.

When charged grains are the dominant ions, their inertia will play important role near the mid plane of the protoplanetary discs. Ion inertia could also be important in transporting angular momentum in accretion discs around compact objects, in cataclysmic variables. For example, in cataclysmic variables, where mass flows from a companion main sequence star on to a white dwarf, the ionization fraction in the disc can vary in a wide range. The ion inertial effect in such a disc could significantly modify the magnetorotational instability and therefore, this instability could be a possible driver of the observed turbulent motion.

Key words: magnetohydrodynamics, star formation, accretion discs, charged grains, magnetorotational instability.

1 INTRODUCTION

Angular momentum transport has long been recognised as a key issue in accretion disc theories (Lynden-Bell 1969; Sakura & Sunyaev 1973). However, until the 1990s, a viable physical mechanism necessary to facilitate this transport in the absence of tidal effects or gravitational instabilities was unknown. The Balbus-Hawley (or magnetorotational) instability (Velikhov 1959; Chandrasekhar 1961) was proposed (Balbus & Hawley 1991; Hawley & Balbus 1992; Balbus & Hawley 1998) as a viable mechanism that can efficiently drive MHD turbulence and transport angular momentum in the disc. This opened the door for its application to a wide variety of astrophysical discs. The requirement for the magnetorotational instability to operate in such a disc is that the ambient magnetic field is subthermal at the disc mid-plane and is well coupled to the disc matter. Although, the lower bound on the weak, subthermal field has never been

specified, in recent work this issue has been addressed in the framework of fully ionized, collisionless cold electron-ion plasma (Krolik & Zweibel 2006). For a highly ionized disc, the requirement of a weak, subthermal field is easily satisfied and the magnetorotational instability grows at the rotation frequency Ω of the disc as a low frequency Alfvén mode with $k V_A \sim \Omega$, where k is the wavenumber and V_A is the Alfvén velocity. However, many astrophysical discs are not well coupled to the magnetic field. Circumstellar, Protoplanetary (PPD), Dwarf Novae (DN), and, proto-neutron-star discs are good examples of weakly ionized discs with very low to low (PPDs) and high (DNs) fractional ionization. In PPDs for example, the sources of ionization are limited to the disc surface and magnetorotational instability may operate only in the outer envelope of the disc (Gammie 1996) unless some nonthermal source of ionization viz. the collision of the energetic electrons with neutrals or magnetorotational instability induced turbulent convective homogeniza-

tion of the entire disc (Inutsuka & Sano 2005) is assumed. DN discs are thought to have both hot and fully ionized accretion state as well as cold and mostly neutral accretion states (Cannizzo 1993; Gammie & Menou 1998). Therefore, the direct application of Balbus & Hawley (1991) results are difficult in a weakly ionized disc.

The effect of non-ideal MHD on the magnetorotational instability has been investigated by several authors: in the ambipolar regime (Blaes & Balbus 1994), hereafter BB94, (MacLow et al. 1995; Hawley & Stone 1998; Kunz & Balbus 2004), the resistive regime (Jin 1996; Papaloizou & Terquem 1997; Balbus & Hawley 1998; Sano et al. 1998; Sano & Miyama 1999; Fleming et al. 2000; Sano et al. 2000; Stone & Fleming 2003) and the Hall regime (Wardle 1999) W99 hereafter; (Balbus & Terquem 2001), BT01 hereafter; (Sano & Stone 2002a,b; Salmeron & Wardle 2003, 2005; Desch 2004). At the densities relevant to cloud cores, ambipolar and Hall diffusion plays an important role in the transport of mass and angular momentum (Wardle & Ng 1999; Balbus & Terquem 2001). W99 and BT01 found that collision of neutrals with the ionized gas in a weakly ionized disc determines the relative importance of Ambipolar, Hall or Ohmic diffusion on the magnetorotational instability. The ambipolar and Hall effects are particularly important when the ionization in the disc is very low and the departure from ideal MHD is severe.

The dynamics of a weakly ionized disc was investigated in the limit of zero inertia of the ionized plasma components by W99 and BT01. This is usually an excellent approximation when the fractional ionization is low, and allows the ionized components of the fluid (viz electrons, ions and grains) to be treated on an equal footing. However, there are situations – even in the low fractional ionisation limit – where the inertia of the charged species is important in determining their drift with respect to the neutral component and hence the diffusion of the magnetic field. In the weakly-ionised limit this becomes important when the inertial terms in the ion equation of motion start to compete with the magnetic and neutral collision terms, in other words when the disc frequency Ω becomes comparable to or exceeds the collision frequency with neutrals *and* the gyrofrequency. This effect may become important for charged dust grains because their large mass implies low collision and gyrofrequencies. For example, in dense PPDs, the dominant ion species are positively charged grains (especially when $\sim n_H \geq 10^{11} \text{ cm}^{-3}$), (Wardle & Ng 1999). In the high fractional ionization regions also, e.g. near the surface of a magnetic cataclysmic variable, ion inertia may become important (Warner 1995).

The effect of ion-inertia on the magnetorotational instability in a two-fluid framework was considered by BB94. The magnetic flux was assumed frozen into the plasma component in their formulation. This limits the applicability of their results to the ambipolar diffusion regime. We know from W99 and BT01 that Hall effect can compete with ambipolar diffusion in the weakly ionized regions of the disc. In the present work we adopt a three component model – neutrals, ions and electrons where by “ion” and “electron” we mean the most massive and least massive charged species, whether positively or negatively charged – to show that ion inertial effects modify the growth rate substantially and increases the parameter window in which the instability may operate. In particular, the growth rate in the presence of ion

inertia may exceed the Oort A value (0.75Ω) limit. Growth rates larger than the Oort A limit have recently been reported for collisionless plasmas in the presence of plasma kinetic/viscous effects (Quataert et al. 2002; Sharma et al. 2003; Balbus 2004), where kinetic and MHD effects combine to give a high growth rate (1.7Ω) and shift the fastest growing modes towards longer wavelengths.

We shall give a general formulation of the problem that will not only cover the regions of applicability of BB94 and W99 but also cover the unexplored regions. This paper revisits the magnetorotational instability in a weakly ionized disc (W99) by incorporating the effect of ion inertia. In section 2, we discuss formulation of BB94 and compare and contrast the region of applicability of BB94, W99 in the context of present work. In section 3, a general formulation of a weakly ionized, near-Keplerian, magnetized disc is given. In section 4 we describe equilibrium and linearization of the disc and derive the dispersion relation. Section 5 discusses the energetics of a weakly ionized, magnetized disc. In this section, we first discuss the role played by the ambipolar and Hall terms in the neutral dynamics. Further, using energy arguments, we discuss the conditions under which ambipolar diffusion can proceed without dissipation and when ambipolar diffusion can destabilize the disc. We also discuss how the Hall term can destabilize the magnetorotational instability. The properties of wave helicity, is derived. In section 6, we give the detailed numerical solution of the dispersion relation. In section 7, application of the result to various astrophysical discs is discussed. Section 8 presents a summary of our results.

2 ION INERTIA AND BB94

BB94 adopted the ambipolar diffusion approximation, in which the magnetic field is frozen into an electrically neutral ionised plasma coupled by collisions to the neutral fluid.

The effect of ion inertia on the dynamics ($\rho_i d_t \mathbf{v}_i$ where ρ_i is the ion mass density and $d_t \equiv d/dt \equiv \partial_t + \mathbf{v}_i \cdot \nabla$ is the convective derivative), is retained in the momentum equation for the ionised component but its effect on Ohm’s law is ignored in BB94. This limits the applicability of the results to the ambipolar diffusion limit.

To better appreciate this point, let us briefly recast the two fluid formulation of BB94 starting with separate ion and electron fluid equations. The equation of motion for the ionised fluid is derived assuming that magnetic field is frozen in the electron fluid,

$$0 = -e n_e (\mathbf{E} + \mathbf{v}_e \times \mathbf{B}/c) \quad (1)$$

and summing the electron and ion momentum equations, to yield

$$\frac{d\mathbf{v}_i}{dt} + \frac{\nabla P_i}{\rho_i} + \nabla \Phi + \nu_{in} (\mathbf{v}_i - \mathbf{v}_n) = \frac{\mathbf{J} \times \mathbf{B}}{c \rho_i} \quad (2)$$

equation (15) of BB94. Here e is the electronic charge, n_e is the electron number density, ν_{in} is the ion-neutral collision frequency, \mathbf{v}_e , \mathbf{v}_i and \mathbf{v}_n are the electron, ion and neutral velocities, Φ is the gravitational potential, P_i is the ion pressure, \mathbf{E} , \mathbf{B} are the electric and magnetic fields and c is the speed of light and $\mathbf{J} = e n_e (\mathbf{v}_i - \mathbf{v}_e)$ is the current density. Taking the curl of (1) and using Maxwell’s equation,

$c \nabla \times \mathbf{E} = -\partial_t \mathbf{B}$, will give $\partial_t \mathbf{B} = \nabla \times (\mathbf{v}_e \times \mathbf{B})$, i.e. the magnetic field is convected away by the electron fluid. If we want to express the right hand side of induction equation in terms of ion velocity, we obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}_i \times \mathbf{B}) - \nabla \times \mathbf{F}_H, \quad (3)$$

where the Hall term is

$$\mathbf{F}_H = \frac{\mathbf{J} \times \mathbf{B}}{e n_e} \quad (4)$$

Since $\mathbf{J} \times \mathbf{B}/c \sim \rho_i (d_t \mathbf{v}_i + \nu_{in} \mathbf{v}_i)$ (here ν_{in} is the ion-neutral collision frequency), the Hall term can be dropped from the induction equation only if $(\nu_{in}, \omega) \ll \omega_{ci} (= eB/(m_i c))$, i.e. the ion-gyration period (ω_{ci}^{-1}) is smaller/ faster than the dynamical time (ω^{-1}) or the ion-neutral collision time (ν_{in}^{-1}). We see that unlike a two component electron-ion plasma, where the Hall term can be introduced only through the ion inertial term ($d_t \mathbf{v}_i$), in a weakly ionized multi-component plasma, the Hall effect appears either via ion-neutral collision or via the $d_t \mathbf{v}_i$ term or both.

Replacing $\partial_t \mathbf{B}$ by $\Delta B/\delta t$ and $\nabla \times \mathbf{F}_H$ by $c B \Delta B/(4\pi n_i e L \Delta x)$, one sees that the Hall term scales as $1/(\omega_{pi} L)$, (here $\omega_{pi} = (4\pi e^2 n_i/m_i)^{0.5}$ is the ion plasma frequency and L is the characteristic size of the system), i.e. the Hall term is important on a scale shorter than the ion-inertial scale. Clearly, Hall MHD introduces two disparate, interacting scales, a microscopic scale, i.e. the ion-skin depth ($\delta_i = c/\omega_{pi} \equiv V_A/\omega_{ci}$) and a macroscopic scale, the disc size. The Hall term can be dropped if ion-inertial effects are unimportant. Leaving out the effect of inertia in the induction equation but retaining them in dynamics is not consistent and, in such a scenario, one would expect that magnetorotational instability will merely shift towards long wavelength, as has already been noted by BB94.

If we start with the ion equation of motion (BB94 (15)), and express the electric field as

$$\mathbf{E} = -\frac{\mathbf{v}_i \times \mathbf{B}}{c} + \frac{m_i}{e} \left[\frac{d\mathbf{v}_i}{dt} + \nabla \Phi + \frac{\nabla P_i}{\rho_i} + \nu_{in} \mathbf{v}_D \right] \quad (5)$$

where $\mathbf{v}_D = \mathbf{v}_i - \mathbf{v}_n$, then taking the curl and using Maxwell's equation, one arrives at the following induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}_i \times \mathbf{B}) - \frac{m_i c}{e} \left[\nu_{in} \nabla \times \mathbf{v}_D + \nabla \times \frac{d\mathbf{v}_i}{dt} \right]. \quad (6)$$

Here uniform density is assumed while operating with the curl on equation (5). This equation has an additional term in comparison with equation (16) of BB94, with important consequences on the magnetic diffusivity, since the rate of change of magnetic flux is given as

$$\begin{aligned} \frac{d}{dt} \int \int \mathbf{B} \cdot d\mathbf{s} &= \int \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint \mathbf{v}_i \times d\mathbf{l} \cdot \mathbf{B} \\ &\equiv \int \int \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v}_i \times \mathbf{B}) \right] \cdot d\mathbf{s}. \end{aligned} \quad (7)$$

Making use of equation (6) in (7), we get

$$\begin{aligned} \frac{d}{dt} \int \int \hat{\mathbf{B}} \cdot d\mathbf{s} &= -\frac{1}{\beta_i} \oint \mathbf{v}_D \cdot d\mathbf{l} \\ &- \frac{1}{\omega_{ci}} \int \int \nabla \times \left[\frac{\partial \mathbf{v}_i}{\partial t} - \mathbf{v}_i \times (\nabla \times \mathbf{v}_i) \right] \cdot d\mathbf{s} \end{aligned} \quad (8)$$

Here $\hat{\mathbf{B}} = \mathbf{B}/B$ and use has been made of $d\mathbf{v}_i/dt = \partial_t \mathbf{v}_i - \mathbf{v}_i \times (\nabla \times \mathbf{v}_i) + \nabla \mathbf{v}_i^2/2$. The ion Hall parameter $\beta_i = \omega_{ci}/\nu_{in}$ gives the ratio between the ion-cyclotron to ion-neutral collision frequencies. The above equation can be rewritten as

$$\frac{d}{dt} \int \int \left[\hat{\mathbf{B}} + \frac{1}{\omega_{ci}} \nabla \times \mathbf{v}_i \right] \cdot d\mathbf{s} = -\frac{1}{\beta_i} \oint \mathbf{v}_D \cdot d\mathbf{l}. \quad (9)$$

We see from equation (9) that the generalized flux that is a combination of magnetic flux and vorticity is not conserved. The rate at which this flux decays is directly related to the collisional coupling between ions and neutrals. If the ion magnetization is weak, i.e. the ion-cyclotron frequency is less than ion-neutral collision frequency ($\beta_i \rightarrow 0$), then the flux-decay rate could be very large for a finite ion-neutral drift speed \mathbf{v}_D . However, if the relative ion-neutral drift is negligible, the generalized flux is conserved irrespective of the ion magnetization level. Thus it is the combination of the magnetic flux and the vorticity that is conserved in the absence of collision ($\beta_i \rightarrow \infty$). The BB94 formulation assumes that magnetic flux is frozen in the ion-fluid which is valid if apart from ignoring the right hand side of equation (9), we also assume that $\omega_{ci} \rightarrow \infty$. In this limit however, the role of ion inertia becomes increasingly unimportant and we approach W99 limit. Clearly, BB94 does not treat the effect of ion inertia in a consistent fashion and their results are not applicable in most of the weakly ionized parameter space where the ion Hall parameter is ~ 1 . In Fig.1, we plot the range of applicability of BB94 and W99. We see that BB94 is applicable when $\beta_i \gg 1$ for $\omega_{ci} > \Omega$ for arbitrary relation between ν_{in} and Ω . BT01 show that $\beta_i \gg 1$ implies that the Hall term dominates over ambipolar term. W99 is applicable for $\omega_{ci} > \Omega$ and $\nu_{in} > \Omega$ for arbitrary β_i . Therefore, $\omega_{ci}/\Omega < 1$ and $\nu_{in}/\Omega < 1$ is an unexplored region in BB94 and W99 framework. We see that in a PPD, for a milliGauss field, for a positive grains of mass $10^{-12} - 10^{-15}$ g, $\omega_{ci}/\Omega < 1$. As has been noted elsewhere, near the mid-plane of PPDs, dust grains can be the dominant charged constituent over extended regions ($\sim 1 - 5$ AU). For sub-micron sized grains $0.1 \mu\text{m}$, negatively charged grains dominate whenever $n_n \gtrsim 10^{11} \text{ cm}^{-3}$ and positively charged grains dominate for $n_n \gtrsim 10^{14} \text{ cm}^{-3}$ (Wardle & Ng 1999). Depending upon neutral density in the disc, the ratio ν_{in}/Ω can have any value and thus, it is important to extend the BB94 analysis to the unexplored regions with full, Hall and ambipolar effects in the spirit of W99. We shall give a general derivation of the induction equation in a weakly ionized medium in the text though, we adopt the conductivity approach of W99 to investigate the effect of ion inertia on the magnetorotational instability and find a significant increase in the growth rate. Further, the parameter window in which instability operates expands considerably to large scale lengths. It will have an important bearing on the angular momentum transport and onset of turbulence.

3 FORMULATION - MHD EQUATIONS

The dynamics of a weakly ionized disc, consisting of electrons, ions, neutrals and charged and neutral dust grains, in the presence of a gravitational field Φ of a central mass point M is described by a set of multifluid equations. A multi-fluid approach representing each and every particle species is not

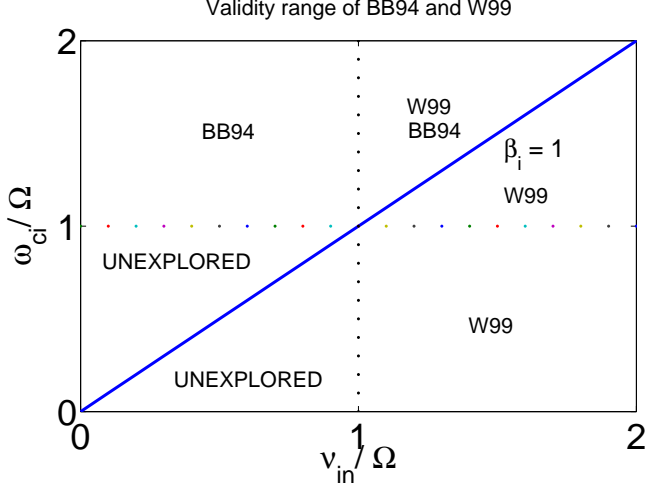


Figure 1. The region of applicability of BB94 and W99.

very fruitful since depending upon the fractional ionization, the presence of some of the ionized component in the disc can be neglected. We shall assume that the ion density in the disc is mainly due to the presence of positively charged grains. Such a situation will correspond to a very dense region of PPDs (Wardle & Ng 1999).

As the disc matter is weakly ionized, generally the inertia of the charged species are neglected (W99). However, apart from pure scientific curiosity about the effect of charged particle inertia on the magnetorotational instability, there are astrophysical environments where the plasma inertial effects may become important. For example, ion inertial effect may compete with collisional and electromagnetic effects in the dense region of PPDs as well as in the high fractional ionization discs around magnetic cataclysmic variables. An estimate of the ion inertial, collisional and electromagnetic effects are given below. Since we wish to investigate the role of ion dynamics on the magnetorotational instability, we shall retain the inertia term in the ion equation of motion. This will result in conductivity tensor σ becoming frequency dependent.

The basic set of equations describing a partially ionized, non-self-gravitating magnetized plasma disc, consisting of electrons, ions, and neutrals, are as follows. The continuity equation is

$$\frac{\partial \rho_j}{\partial t} + \nabla \cdot (\rho_j \mathbf{v}_j) = 0. \quad (10)$$

Here ρ_j is the mass density and \mathbf{v}_j is the velocity of the various plasma components.

The momentum equations for electrons, ions and neutrals are

$$0 = -e n_e \left(\mathbf{E}' + \frac{\mathbf{v}_e \times \mathbf{B}}{c} \right) - \rho_e \nu_{en} \mathbf{v}_e, \quad (11)$$

$$\rho_i d_t \mathbf{v}_i = e n_i \left(\mathbf{E}' + \frac{\mathbf{v}_i \times \mathbf{B}}{c} \right) - \rho_i \nu_{in} \mathbf{v}_i - \rho_i d_t \mathbf{v}_n, \quad (12)$$

$$\rho_n d_t \mathbf{v}_n = -\nabla P - \rho_n \nabla \Phi + \sum_{e,i} \rho_j \nu_{jn} \mathbf{v}_j. \quad (13)$$

Here \mathbf{v}_j is the drift velocity through the neutrals, $\mathbf{F}_j = q_j n_j (\mathbf{E}' + \mathbf{v}_j \times \mathbf{B}/c)$ is the Lorentz force, n_j is the num-

ber density, and, j stands for electrons ($q_e = -e$), and ions ($q_i = e$). Grains are assumed to have single positive electric charge. The electric field $\mathbf{E}' = \mathbf{E} + \mathbf{v}_n \times \mathbf{B}$ is written in the frame comoving with the neutrals. In equation (13), the gravitational potential Φ due to central mass, is given by

$$\Phi = -\frac{GM}{\sqrt{r^2 + z^2}}. \quad (14)$$

At the disc midplane, the Keplerian centripetal acceleration $v_K^2/r \equiv GM/r^2$ balances the radial component of the gravitational potential. Equations (11)-(12) does not have a pressure gradient term since pressure effects will be negligible in a weakly ionized disc. The effects of ionization and recombination are also omitted from the neutral dynamics for the same reason.

The collision frequency is

$$\nu_{jn} = \gamma_{jn} \rho_n = \frac{\langle \sigma v \rangle_j}{m_n + m_j} \rho_n. \quad (15)$$

Here $\langle \sigma v \rangle_j$ is the rate coefficient for the momentum transfer by the collision of the j^{th} particle with the neutrals. The ion-neutral and electron-neutral rate coefficients are (Draine et al. 1983)

$$\begin{aligned} \langle \sigma v \rangle_{in} &= 1.9 \times 10^{-9} \text{ cm}^3 \text{ s}^{-1} \\ \langle \sigma v \rangle_{en} &= 4.5 \times 10^{-9} \left(\frac{T}{30 \text{ K}} \right)^{\frac{1}{2}} \text{ cm}^3 \text{ s}^{-1}. \end{aligned} \quad (16)$$

Adopting a value of $m_i = 30 m_p$ for ion mass and $m_n = 2.33 m_p$ for mean neutral mass where $m_p = 1.67 \times 10^{-24} \text{ g}$ is the proton mass, the ion neutral collision frequency can be written as

$$\nu_{in} = \rho_n \gamma \equiv \frac{m_n n_n \langle \sigma v \rangle_{in}}{m_i + m_n} = 1.4 \times 10^{-10} n_n \text{ s}^{-1}. \quad (17)$$

This also gives the limiting value for very small grains ($\sim 3 - 3000 \text{ \AA}$). For larger, micron sized grains, ion-neutral collision rate can vary between 10^{-10} to 10^{-5} for sizes ranging between a few Angstrom to a few microns. This can be seen if we write the collision rate as (Nakano & Umebayashi 1986)

$$\langle \sigma v \rangle_{in} = 2.8 \times 10^{-5} T_{30}^{\frac{1}{2}} a_{-5}^2, \quad (18)$$

where T_{30} is the gas temperature and a_{-5} is the grain radius in units of 30 K and 10^{-5} cm respectively.

We shall rewrite equations (10)-(13) in the local Keplerian frame. Thus, velocity \mathbf{v} represents the departure from the Keplerian motion; the fluid velocity in the laboratory frame is $\mathbf{v} + \mathbf{v}_K$ and ∂_t is $\partial_t + \Omega \partial_\phi$ in the laboratory frame, where $v_K = \sqrt{GM/r} \hat{\phi}$ is the Keplerian velocity in the canonical cylindrical coordinate system (r, ϕ, z) .

Noting that near the disc midplane, on scales small compared with the disc thickness, the radial gradient in gravitational potential will be exactly cancelled by the centripetal term due to Keplerian motion, and, (r, ϕ) component of the equation (12)-(13) in the absence of any tidal effects, can be rewritten as,

$$\mathbf{A} \mathbf{v}_i = \frac{e}{m_i} \left(\mathbf{E}' + \frac{\mathbf{v}_i \times \mathbf{B}}{c} \right) - \nu_{in} \mathbf{v}_i - \mathbf{A} \mathbf{v}_n, \quad (19)$$

$$\mathbf{A} \mathbf{v}_n = -\frac{\nabla P_n}{\rho_n} + \frac{\mathbf{J} \times \mathbf{B}}{c \rho_n} + O\left(\frac{\rho_i}{\rho_n}\right), \quad (20)$$

where operator $\mathbf{A} = \begin{pmatrix} d_t & -2\Omega \\ 0.5\Omega & d_t \end{pmatrix}$. The induction equation can be written as

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) - c \nabla \times \mathbf{E}' - 1.5 \Omega B_r \hat{\phi}. \quad (21)$$

In (21), $\nabla \times \mathbf{E}'$ contains the effect of non-ideal MHD and the last term accounts for the generation of the toroidal field from the poloidal one due to differential rotation of the disc (W99).

Before describing the conductivity approach and giving a formulation to the problem at hand, we shall give a general derivation of the induction equation starting from equation (11) and estimate the range of applicability of the ion-inertial effects in the accretion discs. Writing $\mathbf{E}' = -\mathbf{v}_e \times \mathbf{B}/c + \eta \mathbf{J}$ as

$$\mathbf{E}' = -\frac{\mathbf{v}_i \times \mathbf{B}}{c} + \eta \mathbf{J} + \mathbf{F}_H. \quad (22)$$

Here

$$\eta = \frac{c^2}{4\pi} \frac{m_e \nu_{en}}{n_e e^2} \equiv \frac{c^2}{4\pi} \frac{m_e n_n <\sigma v>_{en}}{n_e e^2} \quad (23)$$

is the electrical resistivity of the gas and $\mathbf{F}_H = \mathbf{J} \times \mathbf{B}/e n_e$ is the Hall term. Even in the absence of ion inertia - the so called strong coupling approximation, (Shu 1983), the ambipolar term modifies the induction equation, i.e. when $\rho_i \nu_{in} \mathbf{v}_i = \mathbf{J} \times \mathbf{B}/c$,

$$\mathbf{E}' = \eta \mathbf{J} + \mathbf{F}_H - \frac{(\mathbf{J} \times \mathbf{B}) \times \mathbf{B}}{c \rho_i \nu_{in}}. \quad (24)$$

When ion inertial terms are also present,

$$\mathbf{v}_i = \frac{\mathbf{J} \times \mathbf{B}}{c \rho_i \nu_{in}} - \frac{1}{\nu_{in}} \frac{d\mathbf{v}_i}{dt} + O\left(\frac{\rho_e}{\rho_n}, \frac{\rho_i}{\rho_n}\right) \quad (25)$$

and, the generalized Ohm's law becomes

$$\mathbf{E}' = \eta \mathbf{J} + \mathbf{F}_H - \left[\frac{(\mathbf{J} \times \mathbf{B}/c)}{\rho_i \nu_{in}} + \mathbf{A}_1 \mathbf{v}_i \right] \times \mathbf{B}, \quad (26)$$

where $\mathbf{A}_1 = \nu_{in}^{-1} \mathbf{A}$. Now taking the curl of equation (26), one may write the generalized induction equation as

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\mathbf{v} \times \mathbf{B} - \frac{4\pi\eta \mathbf{J}}{c} - \frac{\mathbf{J} \times \mathbf{B}}{e n_e} \right] \\ + \nabla \times \left[\frac{(\mathbf{J} \times \mathbf{B}) \times \mathbf{B}}{c \rho_i \nu_{in}} - (\mathbf{A}_1 \mathbf{v}_i) \times \mathbf{B} \right]. \end{aligned} \quad (27)$$

The induction equation (27) on the right hand side contains inductive, Ohmic diffusion, Hall, ambipolar, and ion inertial terms respectively. The set of equation (10), (12), (13) and (27) can be closed by an equation of state.

Assuming that ion-inertial time scale is of the order of disc rotation period, we may write $\mathbf{A}_1 \sim \Omega/\nu_{in}$. Then the ratio of the ion inertial term to the inductive term can be written as

$$\frac{|\mathbf{v} \times \mathbf{B}|}{|\mathbf{A}_1 \mathbf{v}_i \times \mathbf{B}|} \sim \frac{\nu_{in}}{\Omega}. \quad (28)$$

A different ratio that measures the coupling of the neutral to the ion with respect to the Keplerian frequency have been used by BB94 (Menou & Quataert 2001)

$$Re_A \equiv \frac{\nu_{ni}}{\Omega} = \alpha \frac{\nu_{in}}{\Omega}, \quad (29)$$

where $\alpha = \rho_i/\rho_n$. It is clear from equation (29) that magnetorotational instability can act on both the neutral as well as on the ion fluid simultaneously if $\alpha \sim 1$.

Assuming an equation of state or dropping the pressure gradient term in the neutral equation of motion, equations (10), (19), (20) and (27) with Maxwell's equations

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0, \quad (30)$$

form a complete set.

4 LINEARIZATION

We consider a thin disc implying that the radial scale over which physical quantities vary is much larger than the disc scale height, $H = C_s/\Omega$. The initial steady state is assumed uniform and homogeneous with a vertical magnetic field $\mathbf{B} = B\hat{z}$ and zero \mathbf{v} , \mathbf{v}_n , ∇P , \mathbf{E}' and \mathbf{J} .

We shall assume transverse fluctuations and denote resulting two-dimensional vectors by subscript \perp , to investigate Alfvén modes in the disc. We seek plane wave solution of the form $\exp i(\omega t - k z)$.

4.1 The conductivity tensor

The conductivity tensor σ can be found by considering the drifts of charged particles in response to the electromagnetic field (Cowling 1957; Norman & Heyvaerts 1985; Nakano & Umebayashi 1986; Wardle & Ng 1999). We shall first derive conductivity tensor from equation of motion for charged particles, from (11) and (19) by eliminating \mathbf{v}_e and \mathbf{v}_i in favour of \mathbf{E}' and \mathbf{B} and then give the expressions for parallel, Hall and Pedersen component of this tensor σ in the generalized Ohm's law, $\mathbf{J} = \sigma \cdot \mathbf{E}'$.

Since we assume a homogeneous steady state, it implies that the inhomogeneity scale length (in this case particle scale height H of the disk) is much larger than the characteristic wavelength of the normal modes in the disc. Thus, the contribution of the pressure gradient will be neglected while inverting equation (13). Furthermore, we shall also ignore terms of the order $\sim \rho_i/\rho_n$ in the conductivity tensor. Expressing velocities \mathbf{v}_j in equations (11) and (19) in terms of electric field \mathbf{E}' the relationship between \mathbf{J} and \mathbf{E}' can be written as,

$$\mathbf{J} = \sigma \cdot \mathbf{E}' = \sigma_{\parallel} \mathbf{E}'_{\parallel} + \sigma_H \hat{\mathbf{B}} \times \mathbf{E}'_{\perp} + \sigma_P \mathbf{E}'_{\perp}, \quad (31)$$

where σ_{\parallel} , σ_H and σ_P are the field-parallel, Hall and Pedersen components of the conductivity tensor σ and $\hat{\mathbf{B}}$ is the unit vector along the magnetic field. If charged species j has particle mass m_j , charge $Z_j e$, number density n_j , then the Hall parameter is given as

$$\beta_j = \frac{Z_j e B}{m_j c} \frac{m + m_j}{<\sigma v>_j \rho} \quad (32)$$

where m is the mean neutral particle mass and have dropped subscript 'n' from neutral quantities. As noted in section 2, the Hall parameter determines the magnitude of the magnetic flux transport. The ratio of ion to electron Hall parameter suggest that in the protostellar discs, $\beta_i/\beta_e \sim 10^{-3} \ll 1$. Recall that BB94 formulation is valid when the ion-Hall parameter is large ($\beta_i \gg 1$), i.e. the when Lorentz force

dominates the ion-neutral collisional momentum exchange. However, this limit implies strongly magnetized ions and infrequent collisions.

The conductivity tensor is frequency dependent in the presence of ion inertia. The parallel conductivity is

$$\sigma_{\parallel} = \frac{e c n_i}{B} \left[\beta_e + \frac{\beta_i}{1 + \left(\frac{\omega}{\nu_{in}}\right)^2} - i \beta_i \frac{\frac{\omega}{\nu_{in}}}{1 + \left(\frac{\omega}{\nu_{in}}\right)^2} \right]. \quad (33)$$

Since the plasma is quasi-neutral, we have assumed $n_e = n_i$. The conductivity has become complex. In the low frequency limit, when ion inertial effect is unimportant, i.e. $\omega/\nu_{in} \rightarrow 0$, σ_{\parallel} reduces to W99. In $\omega/\nu_{in} \rightarrow \infty$, σ_{\parallel} has both a real and an imaginary component

$$Re[\sigma_{\parallel}] \simeq \frac{e c n_i}{B} \beta_e, \quad (34)$$

and,

$$Im[\sigma_{\parallel}] \simeq -\frac{e c n_i}{B} \left(\frac{\omega_{ci}}{\omega} \right). \quad (35)$$

The Pedersen conductivity is

$$\sigma_P = \sigma_P^0 \left[1 + \frac{\Delta\sigma_P}{\sigma_P^0} \right]. \quad (36)$$

Here frequency independent part, σ_P^0 is

$$\sigma_P^0 = \frac{e c}{B} \sum_j \frac{n_j Z_j \beta_j}{1 + \beta_j^2}, \quad (37)$$

and frequency dependent part, $\Delta\sigma_P$ is

$$\Delta\sigma_P = \frac{e c}{B} \left(\frac{n_i \beta_i}{1 + \beta_i^2} \right) (Q(\omega) - 1). \quad (38)$$

Here $Q(\omega) = (1 + \beta_i^2) D_1/D_2$, $D_1 = i\omega/\nu_{in} + 1$, $D_2 = D_1^2 + \hat{\Omega}_1 \hat{\Omega}_2$, $\hat{\Omega}_1 = 2\hat{\Omega} + \beta_i$, $\hat{\Omega}_2 = 0.5\hat{\Omega} + \beta_i$ and $\hat{\Omega} = \Omega/\nu_{in}$. In order to isolate real and imaginary part of σ_P and investigate the low and high frequency limits, we write $Re[Q(\omega)]$ and $Im[Q(\omega)]$ as

$$\frac{Re[Q(\omega)]}{1 + \beta_i^2} = \frac{1 + \left(\frac{\omega}{\nu_{in}}\right)^2 + \hat{\Omega}_1 \hat{\Omega}_2}{\left[1 - \left(\frac{\omega}{\nu_{in}}\right)^2 + \hat{\Omega}_1 \hat{\Omega}_2\right]^2 + 4 \left(\frac{\omega}{\nu_{in}}\right)^2}, \quad (39)$$

and,

$$\frac{Im[Q(\omega)]}{1 + \beta_i^2} = -\frac{\left(\frac{\omega}{\nu_{in}}\right) \left[1 + \left(\frac{\omega}{\nu_{in}}\right)^2 - \hat{\Omega}_1 \hat{\Omega}_2\right]}{\left[1 - \left(\frac{\omega}{\nu_{in}}\right)^2 + \hat{\Omega}_1 \hat{\Omega}_2\right]^2 + 4 \left(\frac{\omega}{\nu_{in}}\right)^2} \quad (40)$$

In the low frequency limit, $\omega/\nu_{in} \rightarrow 0$, and assuming $\omega \sim \Omega$, $Re[Q(\omega)] \simeq 1$ and $Im[Q(\omega)] \simeq 0$. Thus $\sigma_P = \sigma_P^0$. In the high frequency limit, when $\omega/\nu_{in} \rightarrow \infty$, $Re[Q(\omega)] \approx 0$ and $Im[Q(\omega)] \approx -\nu_{in}/\omega$. Thus,

$$Re[\sigma_P] = \frac{e c}{B} \frac{n_e \beta_e}{1 + \beta_e^2}, \quad (41)$$

and,

$$Im[\sigma_P] = -\frac{e c}{B} n_e \beta_i \left(\frac{\omega}{\nu_{in}} \right)^{-1}. \quad (42)$$

Like the parallel conductivity, the real part of the Pedersen conductivity in the high frequency limit is mainly due

to electron magnetization and imaginary part is due to ion magnetization. Complex resistivity is well known in LCR circuits where the resonance condition is found by setting imaginary part of the impedance to zero. In the present case, a similar resonance condition can be derived by setting numerator of equation (40) to zero, i.e. $\omega^2/\nu_{in}^2 = -1 + \hat{\Omega}_1 \hat{\Omega}_2$, we get

$$1 + \frac{\Delta\sigma_P}{\sigma_P^0} \simeq \frac{\hat{\Omega}_1 \hat{\Omega}_2}{2 \left(1 + \frac{\omega^2}{\nu_{in}^2}\right)}. \quad (43)$$

It is important to note that the scale of frequency dependent conductivity associated with the resonance ($i\omega \sim \Omega$), can become larger than the dc conductivity. We anticipate, therefore, that the frequency dependent Pedersen conductivity will significantly modify the magnetorotational instability.

The Hall conductivity is

$$\sigma_H = - \begin{pmatrix} 1 & \frac{\Delta\sigma_{Hr}}{\sigma_H^0} \\ \frac{\Delta\sigma_{H\phi}}{\sigma_H^0} & 1 \end{pmatrix} \begin{pmatrix} \sigma_H^0 \\ \sigma_H^0 \end{pmatrix} \quad (44)$$

where the dc part is given as

$$\sigma_H^0 = \frac{e c n_i}{B} \sum_j \frac{Z_j}{1 + \beta_j^2}. \quad (45)$$

The frequency dependent parts, $\Delta\sigma_{Hr}$, and, $\sigma_{H\phi}$ are

$$\begin{aligned} \Delta\sigma_{Hr} &= \frac{e c}{B} \frac{n_i \beta_i^2}{1 + \beta_i^2} (H_1(\omega) - 1), \\ \Delta\sigma_{H\phi} &= \frac{e c}{B} \frac{n_i \beta_i^2}{1 + \beta_i^2} (H_2(\omega) - 1), \end{aligned} \quad (46)$$

and,

$$H_j(\omega) = \frac{(1 + \beta_i^2) \hat{\Omega}_j}{\beta_i D_2}, \quad (47)$$

for $j = 1, 2$. The radial and azimuthal component of the Hall conductivities are not equal due to the unequal radial and azimuthal coefficient in the ion momentum equation. The real and imaginary part of H_j is given by equations (39)-(40) if we recognize that right hand side of equation (47) has a factor $1/D_2 = Q(\omega)/D_1 (1 + \beta_i^2)$. Thus, near resonance

$$\left(1 + \frac{\Delta\sigma_{Hr}}{\sigma_H^0}\right) \simeq \frac{\hat{\Omega}_1 (1 + \hat{\Omega}_1 \hat{\Omega}_2)}{\beta_i \left[(1 + \hat{\Omega}_1 \hat{\Omega}_2)^2 + 4 \left(\frac{\omega}{\nu_{in}}\right)^2 \right]}. \quad (48)$$

Except for $\hat{\Omega}_1$ becoming $\hat{\Omega}_2$, the remainder of the expression for the azimuthal factor will be identical to equation (48).

The analogy to LCR resonance can be brought closer if we express Ohm's law, equation (31) in diagonal form. To that end, we shall express \mathbf{E}'_{\perp} in the eigen-basis vectors of the rotation operator $\hat{\mathbf{e}}_{\pm} = (\mathbf{e}_x \pm i \mathbf{e}_y)/\sqrt{2}$. Then $\hat{\mathbf{B}} \times \hat{\mathbf{e}}_{\pm} = \mp i \hat{\mathbf{e}}_{\mp}$ and, Ohm's law for the transverse component \mathbf{J}_{\pm} can be written as

$$\mathbf{J}_{\pm} = (\sigma_P \mp i \sigma_H) \mathbf{E}'_{\pm\pm}. \quad (49)$$

While writing equation (49), we have assumed $\sigma_{Hr} = \sigma_{H\phi} = \sigma_H$. We may write $\mathbf{J}_{\pm} = Z_{\pm}(\omega) \mathbf{E}'_{\pm\pm}$. Here the impedance $Z_{\pm}(\omega) = R_{\pm} + i X_{\pm}(\omega)$, with $R_{\pm} = Re[\sigma_P] \pm Im[\sigma_H]$, and, $X_{\pm} = Im[\sigma_P] \mp Re[\sigma_H]$. Near resonance, $Z_{\pm} = R_{\pm}$. Thus

$$\mathbf{J}_{\pm} = R_{\pm} \mathbf{E}'_{\perp \pm} \propto \frac{\mathbf{E}'_{\perp \pm}}{\left(1 + \frac{i\omega}{\nu_{in}}\right) \left(1 - \frac{i\omega}{\nu_{in}}\right)}. \quad (50)$$

For a growing mode $i\omega \sim \Omega$, and, oscillations in the current can take place in the absence of a neutral-frame electric field $\mathbf{E}'_{\perp} \rightarrow 0$. In a weakly ionized disc, if $\beta_i \lesssim 1$, the oscillation in the current is set by the ions diffusing across the ambient magnetic field. For $\nu_{in} \sim i\omega$, near resonance, collision will act like a driver of the resonance.

We note that near resonance, the conductivity may change sign. From (43), when $\Omega < \nu_{in}$, the negative conductivity will play an important role. The ratio of the dynamical to the ion-neutral collision frequency determines whether negative conductivity is important. The DC conductivity becoming negative within certain frequency range, in the microwave irradiation is well known in the condensed matter literature, e.g. Ryzhii (2005).

4.2 Dispersion relation

The linearized neutral equation of motion (20) can be written as

$$\begin{pmatrix} \frac{\omega}{-i\Omega} & 2i\Omega \\ -\frac{i\Omega}{2} & \omega \end{pmatrix} \delta \mathbf{v}_{\perp} = -k v_A^2 \begin{pmatrix} \delta \mathbf{B}_{\perp} \\ B \end{pmatrix}, \quad (51)$$

where subscript \perp denotes two dimensional vector in the disc plane and $v_A = B/\sqrt{4\pi\rho}$ is the Alfvén velocity in the total fluid. The linearized induction equation, after substituting for $\delta \mathbf{v}_{\perp}$ is given as

$$\begin{pmatrix} \omega_A^2 + 3\Omega^2 & 2i\omega\Omega \\ -2i\omega\Omega & \omega_A^2 \end{pmatrix} \delta \mathbf{B}_{\perp} = ik \begin{pmatrix} 2\Omega & i\omega \\ -i\omega & \Omega \end{pmatrix} \delta \mathbf{E}'_{\perp}, \quad (52)$$

where $\omega_A^2 = \omega^2 - k^2 v_A^2$.

In the ideal MHD limit, when $\delta \mathbf{E}'_{\perp} = 0$, one recovers magnetorotational mode. In the absence of rotation, a dispersion relation for ideal MHD can be derived by setting determinant of left hand side matrix to zero. The departure from ideal MHD is due to the collisional effects. They will appear when electric field is eliminated in favour of magnetic field utilizing Ohm's law and Maxwell's equation.

Making use of linearized Ampere's law

$$\delta \mathbf{J}_{\perp} = \frac{ikc}{4\pi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta \mathbf{B}_{\perp}, \quad (53)$$

in the generalized Ohm's law,

$$\delta \mathbf{E}'_{\perp} = \frac{-ikc}{4\pi\Delta(\omega)} \begin{pmatrix} s\sigma_{Hr} & -\sigma_P \\ \sigma_P & s\sigma_{H\phi} \end{pmatrix} \delta \mathbf{B}_{\perp}, \quad (54)$$

where $\Delta(\omega) = \sigma_{Hr}(\omega)\sigma_{H\phi}(\omega) + \sigma_P(\omega)^2$ and, $s = \text{sign}(B_z)$. Introducing normalized variable $y = i\omega/\Omega$ frequency dependent part of the conductivities can be written as,

$$\begin{aligned} Q(y) &= (1 + \beta_i^2) F(y), \\ H_1(y) &= \frac{(1 + \beta_i^2) \hat{\Omega}_1 F(y)}{D_1 \beta_i}, \\ H_2(y) &= \frac{(1 + \beta_i^2) \hat{\Omega}_2 F(y)}{D_1 \beta_i}, \end{aligned} \quad (55)$$

where $F(y) = D_1(y)/D_2(y)$ and $D_1(y) = 1 + \hat{\Omega}y$, $D_2 = D_1^2 + \hat{\Omega}_1 \hat{\Omega}_2$, $y = i\omega/\Omega$.

Eliminating $\delta \mathbf{E}'_{\perp}$ from equation (52), one gets the following dispersion relation

$$a \left(\frac{k v_A}{\Omega} \right)^4 + b \left(\frac{k v_A}{\Omega} \right)^2 + c = 0 \quad (56)$$

$$a = \chi^2 G(y)^2 + \chi G(y) (2s\sigma_{H\phi} + 0.5s\sigma_{Hr} + 2y\sigma_P) \sigma_{\perp}^{-1} + (y^2 + 1) G(y), \quad (57)$$

$$b = (2y\sigma_P - 1.5s\sigma_{H\phi}) (y^2 + 1) \chi G(y) \sigma_{\perp}^{-1} + \chi^2 G(y)^2 (2y^2 - 3), \quad (58)$$

$$c = \chi^2 G(y)^2 y^2 (y^2 + 1). \quad (59)$$

Here $G(y) = \Delta(y)/\sigma_{\perp}^2$, $\sigma_{\perp} = \sqrt{\sigma_P^2 + \sigma_H^2}$ and the parameter $\chi = \omega_c/\Omega$ is the normalized critical frequency $\omega_c = 4\pi(v_A/c)^2 \sigma_{\perp}$ similar to W99. As has been noted in W99, the ideal MHD description is valid in the large χ limit. When $k v_A \geq \omega_c$ non-ideal MHD effect become important.

When ion inertial effects are ignored, the expressions for a , b and c , in equations (57)-(59) reduces to W99. Ideal MHD is recovered in $\chi \rightarrow \infty$ limit. As has been discussed in W99, the Hall term has considerable effect on the magnetorotational instability growth rate ($\sim \Omega$) in the small χ limit. This is because the Hall effect and collisions are intricately linked in a partially ionized plasma. Since the Hall effect is due to ion-neutral collision, small wavelength fluctuations are suppressed in the Hall regime and only long wavelength fluctuations will grow. We shall see that both Hall as well as Ambipolar diffusion will become important in the small χ , high frequency limit, suggesting that the presence of ion inertial effect destabilizes the weakly ionized disc at all wavelengths.

5 ENERGETICS OF THE DISC

Before we discuss numerical results, let's examine the various factors (viz. Lorentz force, Joule heating) that may affect magnetorotational instability. The Lorentz force, $\mathbf{J} \times \mathbf{B}$, acts on the neutrals through collisions. Making use of equation (31), it can be written as

$$\frac{\mathbf{J} \times \mathbf{B}}{c\rho\Omega} = (\chi_P \mathbf{V}_B + \chi_H \hat{\mathbf{B}} \times \mathbf{V}_B). \quad (60)$$

Here $\chi_P = (\sigma_P/\sigma_{\perp})\chi$, $\chi_H = (\sigma_H/\sigma_{\perp})\chi$ and $\mathbf{V}_B = c\mathbf{E}' \times \mathbf{B}/B^2$ is the drift velocity of \mathbf{B} through the neutrals. The first term on the right hand side is a measure of simultaneous acceleration and frictional drag; viewed from a neutral frame, this force accelerates the neutral towards $\mathbf{E}' \times \mathbf{B}$ velocity. The parameter χ_P provides the strength of the collisional coupling. With decreasing χ_P , i.e. when the ionized medium is far from ideal MHD regime, this term may become increasingly unimportant. Thus the modification to the ideal MHD modes will be severe in the small χ_P limit since collision modifies the fluid response to the magnetic tension. Note that in the ambipolar regime, this term is responsible for dissipation as well as feeding of energy to the neutrals. The second term will accelerate the neutral in the direction of \mathbf{E}' .

In order to understand the implications for the magnetorotational instability, we need to identify the conditions under which energy is fed to the fluctuations. Recall that the electric field $\mathbf{E}' = \mathbf{E} + \mathbf{v}_n \times \mathbf{B}/c$ is given in the neutral

frame and thus,

$$\mathbf{J} \cdot \mathbf{E} = \sigma_{\parallel} \mathbf{E}_{\parallel}'^2 + \sigma_P \left(\mathbf{E}_{\perp}'^2 + \mathbf{v}_n \cdot \mathbf{E}_{\perp}' \times \mathbf{B}/c \right) + \frac{B}{c} \sigma_H (\mathbf{v}_n \cdot \mathbf{E}_{\perp}') . \quad (61)$$

Clearly then, the energy exchange consists of Joule heating and acceleration of the neutral medium. The term $\sigma_{\parallel} \mathbf{E}_{\parallel}'^2 + \sigma_P \mathbf{E}_{\perp}'^2$ is the Joule heating. This term is always positive for positive σ_P . However, since σ_P may become negative near resonance and the Ohmic term $\sigma_P \mathbf{E}_{\perp}'^2$ may feed rather than dissipate energy. Therefore, in the ambipolar regime, fluctuations may grow. The terms $\mathbf{v}_n \cdot \mathbf{E}_{\perp}' \times \mathbf{B}$ and $\mathbf{v}_n \cdot \mathbf{E}_{\perp}'$, for ambipolar and Hall respectively, corresponds to the feeding or, extraction of the kinetic energy by the Lorentz force. Therefore, depending upon the sign of the kinetic energy terms, the Lorentz force may facilitate either growth or damping of the magnetorotational instability.

6 RESULTS

We shall solve the dispersion relation (56) numerically in various limiting cases and discuss modifications due to ion inertia. In the absence of ion inertia, various χ limits and its effect on the magnetorotational instability have been discussed in detail in W99. We assume that electrons are frozen in the magnetic field, i.e. $\beta_e = \infty$. In this limit, $\sigma_P^0/\sigma_H^0 = \beta_i$. Therefore, we shall solve the dispersion relation (56) by varying key parameters χ , β_i and $\nu = \nu_{in}/\Omega$. In accretion disc environment, the value of β_i , may vary in a wide range. we shall choose β_i between 0.1 and 1. Although higher value of β_i can be chosen, the growth rate of magnetorotational instability will be very small. The parameter ν_{in}/Ω is similarly varied in a wide range.

6.1 Variation of β_i for $\chi = 0.1$, $\nu = 1$

In Fig. 2(a) we plot the growth rate by varying β_i while keeping $\nu = 1$ and $\chi = 0.1$ fixed. With the decreasing β_i , the parameter window of magnetorotational instability extends towards short wavelength and the growth rate exceeds ideal MHD limit for $\beta_i = 0.1$. With the decrease in β_i when $\omega_{ci} < \nu_{in} \sim \Omega$, the mode grows upto 0.92Ω for $\beta_i = 0.1$. This can be possibly attributed to the fact that owing to the faster collisional (and Keplerian) time scales in comparison with the gyration time, rotational free energy becomes available to the fluctuations at the collisional time scale. For further decrease in the value of β_i to 0.01, the maximum growth rate decreases. This behaviour indicates that if β_i is increased beyond some critical value, the Hall diffusion is dominated by the ambipolar diffusion. The growth rate for $\beta_i = 1$ curve is small. This regime correspond to $\omega_{ci} \sim \nu_{in} \sim \Omega$, i.e. the rate of ion gyration and collision with the neutrals is comparable with the rotational frequency. With the increase of β_i , the growth rate decreases and disappears altogether for very large β_i . Thus, Hall effect (caused by the ion inertial and collisional effects), starts dominating the ambipolar diffusion and the mode starts growing. Further decrease in β_i and increase in Hall conductivity reduces the growth rate to 0.82Ω . The collisional effect weakens with decreased β_i and thus, the parameter window operates in both small and large wavelength regimes.

6.2 Variation of χ and ν for fixed β_i

In Fig. 2(b) the growth rate is given for varying χ and fixed $\beta_i = 0.1$, $\nu = 1$. With the decreasing χ (i.e. increasing collisional coupling), the wavenumber window of the instability shifts towards long wavelength, consistent with the fact that non-ideal MHD effects start playing an increasingly important role for smaller χ . The result is similar to W99. This result is also in agreement with Blaes & Balbus (1994). We see that with the decreasing χ the growth rate remains unchanged. Only change is in the wavenumber window that shifts towards the long wavelength consistent with the Hall dominated result of W99.

In fig. 2(c) we plot the growth rate for $\chi = 1$ and $\beta_i = 0.1$ for varying ν . The results are along the expected line. With the increase in collision, the growth rate decreases due to increased role of dissipation and the parameter window of instability shifts towards long wavelength. When the ion-neutral collision rate is comparable (or smaller) to the rotational frequency, the MRI is unaffected by the collision. This is because the rate of dissipative loss of the energy is comparable or slower than the rotational time scale $\nu_{in}^{-1} \sim \Omega^{-1}$. Thus dissipation stops affecting the growth rate and it saturates around ~ 0.75 . Further decrease of ν does not affect the growth rate.

6.3 General limit with $\beta_i = 1$

In figure 3(a) we plot the growth rate for the positive orientation of the magnetic field with respect to the rotation axis, i.e. $s = 1$. The results are similar to the known results of W99. However, there are some interesting differences towards small χ limit. Whereas, the wavenumber window of magnetorotational instability shifts towards longer wavelengths in small χ limit, the growth rate of instability is not very sensitive. The ion inertia is able to provide the free energy to the fluctuations that can counterbalance the dissipative losses in small χ regime where collision coupling between ion and neutral is very strong. Thus, with the decrease in the value of χ , small wavelength fluctuations are all suppressed leaving large wavelength modes to grow at $\sim 0.82\Omega$.

In figure 3(b) we plot the growth rate against ν_{in}/Ω . We see that when $\nu_{in} \geq \Omega$ i.e., when collision time $t_c \equiv \nu_{in}^{-1}$ is smaller than the rotational time $t_r \equiv \Omega^{-1}$, and the free energy is dissipated by the collision. The growth rate of fluctuations decreases. We see from the plot that with increasing ν the magnetorotational instability growth rate reduces significantly. However, the growth rate becomes insensitive beyond $\nu = 10$. This indicates that in the large ν limit, ion inertial effect, namely Hall effect starts cancelling dissipation and thus, growth rate becomes insensitive to any further increase of ν . In the opposite limit, i.e. when $\nu \leq 1$ (or, $t_r < t_c$), the energy available to the fluctuation is at the rotational time scale and, infrequent, slow ion-neutral collision is unable to influence the growth of the instability. At $\nu = 0.01$, the growth rate becomes maximum $\sim 0.9\Omega$ and any further decrease in ν do not change the growth rate significantly.

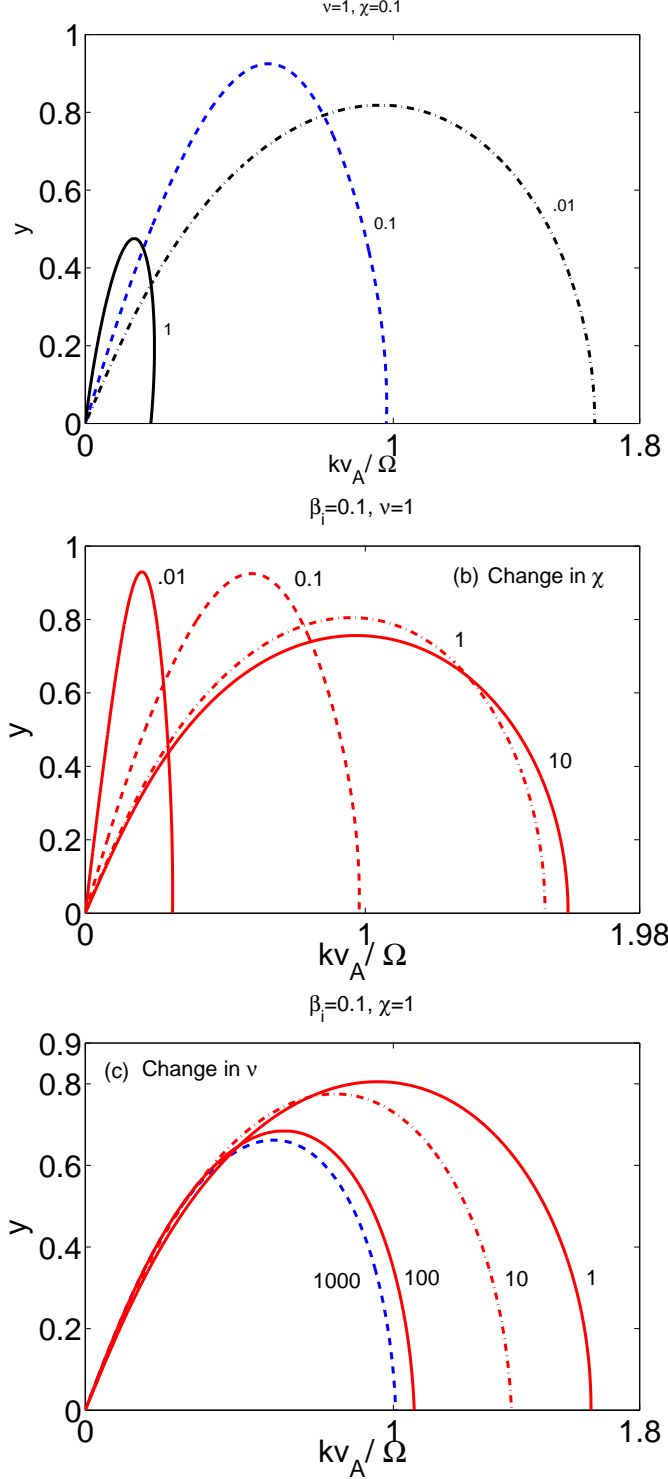


Figure 2. The magnetorotational instability growth rate for $\sigma_P = 1$ for varying β_i and σ_H with fixed $\nu = 1$ and $\chi = 0.1$ is plotted against the wavenumber. The number against the curve is the value of ion Hall β_i for $\sigma_H = 1, 10$ and 100 . In figure 2(b) we hold ν fixed and give growth rate for various ν . In Fig.2(c) we hold χ fixed and vary ν .

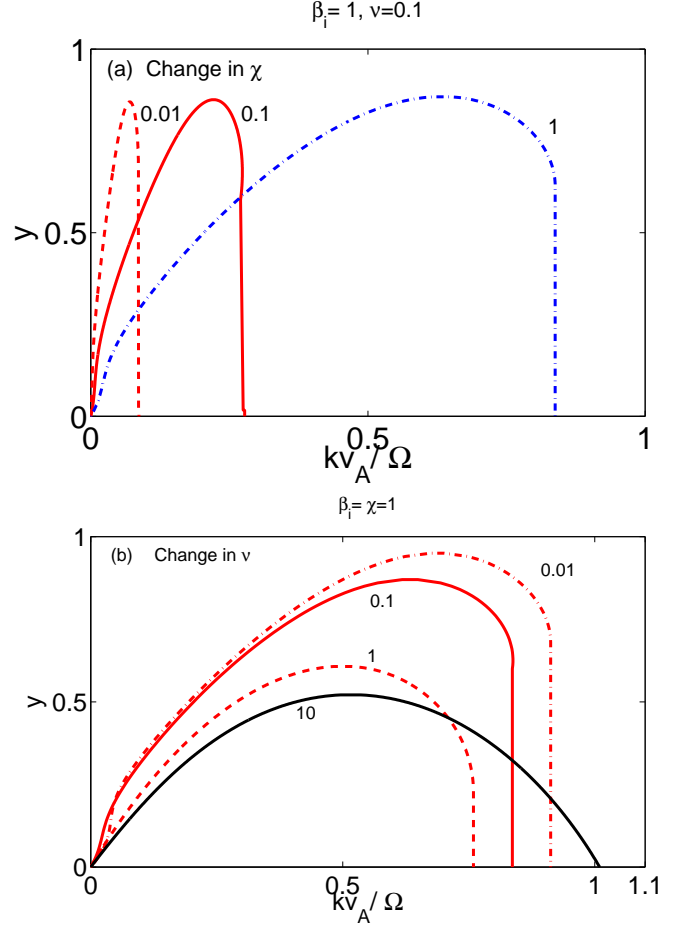
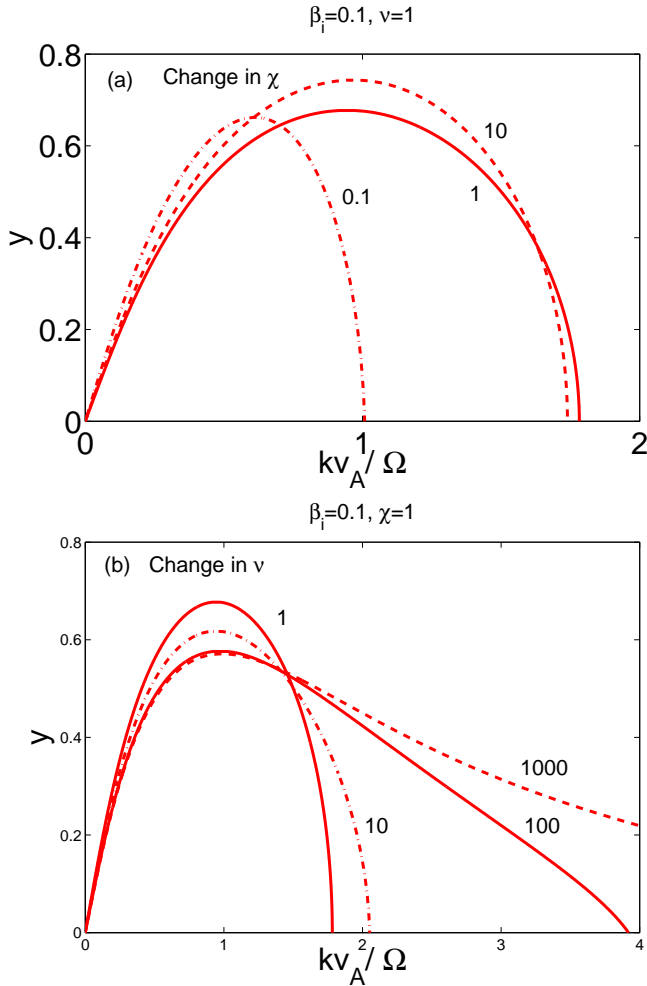


Figure 3. As for Fig.2 but for $\sigma_P = \sigma_H = 1$

6.4 Weak ambipolar limit with $\beta_i = 0.1$, $s = -1$

In figure 4(a) and 4(b) we plot the growth rate for $B_Z < 0$ by varying χ and ν respectively. In figure 4(a) the growth rate is slightly smaller than for corresponding case in Fig. 3(a) for $s=1$. For $\chi \leq 0.1$ the growth rate is insensitive to the changing value of χ implying that if collision frequency is an order of magnitude smaller than the Keplerian frequency, the ambipolar effects are entirely compensated by the Hall and the growth rate remains constant. The curves are similar to W99 except that the mode exists for much smaller value of χ than was the case in W99. Also, the growth rate is larger. The growth rate increases with increasing χ and attains maximum value for $\chi = 10$. If χ is increased further, there is no change in the growth rate. The rate at which rotational energy becomes available and dissipation operates become comparable. For $\chi > 1$, the ideal MHD limit is approached and thus the effect of collision diminishes. Thus for $\chi = \infty$ the growth rate is maximum.

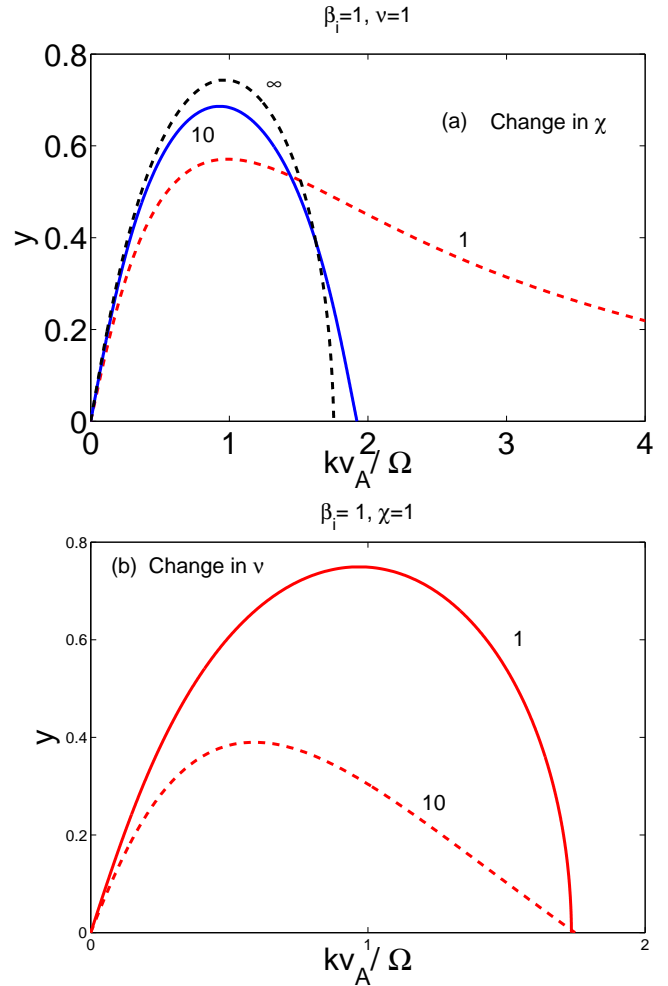
In figure 4(b) the growth rate is not very sensitive to change in ν except when it becomes large. For large ν the growth rate reduces in comparison with the small ν values and parameter window extends towards smaller wavelength.

Figure 4. As for Fig. 2(b), 2(c) but for $s = -1$.

6.5 General limit for $s = -1$ and $\beta_i = 1$

In Fig. 5(a) we plot the growth rate for $s = -1$ for varying χ . The growth rate is insensitive when $\chi < 1$. Small χ is a measure of departure from ideal MHD and we note that Hall effects play important role for $\chi < 1$. Since for $s = -1$ the sign of the wave helicity $\mathbf{\Omega} \cdot \delta \mathbf{B}$ is negative since Alfvén wave is propagating in the negative direction (??). Thus, the increase in the non-ideal effect, manifested through Hall terms, does not have any bearing on the growth rate. With the increase of χ the magnetorotational instability growth rate approaches ideal limit.

In Fig. 5(b) the variation of growth rate with ν is given. For $\nu \leq 1$ there is no change in the growth rate. For $\nu > 1$ the instability is damped due to dissipation. The sign of the helicity ensures that non ideal effect do not feed the energy to the fluctuations. Thus we see the decrease in the growth rate with increasing ν . Therefore, we see that in the general case when the ambipolar conductivity may change, the growth rate of the instability becomes larger than the rotation frequency and the instability operates at long wavelengths. Thus, the ion-inertial effect introduces entirely new feature to the dynamics of a weakly magnetized disc. Not only it changes the length scale over which the instability can operate but also, how fast it can operate. These feature

Figure 5. As for Fig.2 but with $s = -1$.

makes ion inertial effect very important for the application to the protostellar discs.

7 APPLICATIONS

The modification to the magnetorotational instability by ion inertia may have wide ranging application in the astrophysical discs. Before discussing the application of the results, we shall note that the parameters χ , ν and β_i are not independent but are related by the following equation

$$\frac{\rho_i}{\rho_n} = \sqrt{1 + \beta_i^{-2}} \left(\frac{\chi}{\nu} \right), \quad (62)$$

where $\nu \equiv \nu_{in}/\Omega$ is the normalized collision frequency and χ is the measure of non-ideal MHD effects. Therefore, the choice of χ , β_i and ν constrains the ratio ρ_i/ρ_n and hence, the level of fractional ionization.

At densities relevant to cloud cores and protostellar discs (densities $\geq 10^{11} \text{cm}^{-3}$), grains are the dominant charge carriers and their presence can significantly alter the dynamics of the disc. The ionization fraction is strongly affected by the abundance and size distribution of the grains through the recombination process on the grain surface. Near the midplane of the PPDs grains are the main charged

constituent (Wardle & Ng 1999). We shall assume a Keplerian frequency for the minimum mass solar nebula ($0.1 M_{\odot}$), $\Omega \sim 10^{-8} \text{ s}^{-1}$. Then, from equations (15) and (18) we write

$$\frac{\nu_{in}}{\Omega} \sim \frac{m_n}{m_i} 10^{14} a_{-5}^2 \left(\frac{n_n}{10^{11} \text{ cm}^{-3}} \right). \quad (63)$$

Taking $m_n/m_i \sim 10^{-14} - 10^{-15}$, we see that ν_{in} is comparable to the dynamical frequency and ion-inertia becomes important. The collision of the energetic electron with neutrals or magnetorotational instability induced turbulent convective homogenization of the entire disc (Inutsuka & Sano 2005) may allow the magnetic field to couple to the disc matter near the mid plane. Therefore, the ion-inertia may modify the parameter window of instability near the mid-plane of the disc.

In AGNs, for example NGC 4258, a thin disc of 0.2 pc diameter, bound by a central mass of $\sim 2.1 \times 10^7 M_{\odot}$, is rotating with a velocity 900 km s^{-1} (Greenhill et al. 1995). The observed emission emanates from an annulus of inner radius $\sim 0.13 \text{ pc}$ and outer radius of 0.25 pc . Taking $R = 0.1 \text{ pc}$, we get $\Omega \sim 10^{-10} \text{ s}^{-1}$. Thus the ratio $\nu_{in}/\Omega \sim n_n/\text{cm}^{-3}$. Taking ionization fraction $X_e = 10^{-5}$ (Menou & Quataert 2001) at 0.1 pc , we see from equation (29) that $Re_A = 10^{-3} n_n/\text{cm}^{-3}$. The neutral density $n_n \sim 10^7 \text{ cm}^{-3}$ and thus, both ion-neutral ν_{in} as well as neutral-ion ν_{ni} collision frequencies are very large in comparison with the rotation frequency. The charged grains are negligible in such a disk since from $\rho_i/\rho_n = 10^{-2}$, we have $n_i/n_n \sim 10^{-14}$ for micron-sized grains. Therefore, the charged grains are absent in such a disk and grain inertia will have no effect on the disc dynamics.

Given the uncertain nature of the disc size, if we assume a disc of 100 pc with a temperature gradient towards the outer edge of the disc then the inertial effects in such a disc will be due to the charged grains near the core of the disc and due to the lighter ions near the surface region of the disc. For $\Omega \sim 10^{-8} \text{ s}^{-1}$ (at 100 pc), $\nu_{in}/\Omega \sim 10^{-2} n_n/\text{cm}^{-3}$, $X_e \leq 1$, we see that in the surface region of an disc, both ion and neutral fluid will be affected by the magnetorotational instability. Thus, ion-inertial effect may be important in exciting MHD turbulence in the whole disc.

Cataclysmic Variables (CV) are close binary systems with a white dwarf accreting material from a Roche-lobe and a companion low mass main sequence star. The typical orbital frequency of the CVs vary between $\Omega \sim 10^{-3} - 10^{-5} \text{ s}^{-1}$. Then $\nu_{in}/\Omega \sim (10^{-5} - 10^{-7}) \times n_n/\text{cm}^{-3}$. For $n_n \sim (10^5 - 10^7) \text{ cm}^{-3}$, $\nu_{in}/\Omega \sim O(1)$. The temperature in CVs may vary in a wide range and disc can be modeled either as a weakly ionized plasma (Gammie & Menou 1998) or a completely ionized plasma (Saxton et al. 2005). Clearly, both ion and neutral inertial effect operate on an equal footing in CVs.

The circumnuclear disc at the galactic centre has a typical constant rotation speed of 110 Km s^{-1} ((Genzel & Townes 1987)) between 2 to 4 pc . The corresponding rotational frequency at $2 - 4 \text{ pc}$ is, $\Omega \sim 10^{-12} - 10^{-13} \text{ s}^{-1}$. Then the ratio between ion-neutral collision to the Keplerian rotation frequency is

$$\frac{\nu_{in}}{\Omega} \approx (10^2 - 10^3) \times n_n, \quad (64)$$

for given ν_{in} value (equation (17)). Hence at $2 - 4 \text{ pc}$, the ion inertial response time, Ω^{-1} is thousand times slower than

the collisional momentum exchange time, ν_{in}^{-1} . Therefore the ability of ion inertia to modify the magnetorotational instability at 2 pc is unclear although $Re_A \sim 1$ for $n_i/n_n \sim 10^{-3}$. At 100 pc where Ω drops by two orders, ν_{in} becomes comparable to the rotational frequency and inertial effect on the magnetorotational instability may become important.

8 CONCLUSIONS

The paper examines the role of ion inertial effect on the magnetorotational instability in a weakly ionized, thin, magnetized Keplerian disc. The vertical stratification and radial and azimuthal variations were neglected - an approximation valid for the wavelengths small compared to the disc scaleheight. The conductivity tensor becomes time dependent in the presence of ion inertial terms. This may result in the conductivity becoming negative near resonance. Further, radial and azimuthal component of Hall conductivity will be different. The following results were found.

(i) The conductivities in a weakly ionized gas is in general complex in the presence of ion inertia and may become negative near the resonance points. The magnetorotational instability gets significantly modified in the presence of time dependent conductivities.

(ii) In weak ambipolar regime, the presence of ion-inertial effect substantially modifies the behaviour of the instability in the non-ideal ($\chi < 1$) limit. The maximum growth rate in the Hall dominated regime is ~ 0.92 (in the units of Ω) and large wavelength fluctuations can grow due to the inertial effect.

For a fixed χ , the growth rate is maximum (~ 0.8) for $\nu = 1$ and starts to decrease with increasing ν . Further, the parameter window shifts towards longer wavelength with increasing ν .

(iii) When both ambipolar and hall diffusion are comparable, the maximum growth rate is ~ 0.85 and 0.95 for fixed ν and χ respectively. The increase in the ambipolar diffusion causes the fixed χ case to have larger growth rate than fixed ν case.

To summarize, it is a common feature that large scale fluctuations exhibit the maximum growth rate of the magnetorotational instability when the ion inertial effects are included in the dynamics. This may have important implication on the onset of turbulence in weakly ionized discs. For example, in PPDs, when grains are dominant ions, the grain inertial effect will significantly modify the magnetorotational instability growth rate and thus, will play an important role in the onset of hydromagnetic turbulence. In AGNs also, grain will play important role. In CVs grain will provide the ion inertia away from the surface of the dwarf novae whereas, lighter ionized elements will provide the inertial effect close to the surface of the disc. Therefore, in CV discs, inertial effect may be important all across the disk and inertia modified magnetorotational instability may effect the whole disc. In circumnuclear discs, the effect of ion inertia may be important far away from the centre of the disc. All in all, ion inertia seems to play an important role on the onset of magnetorotational instability.

Present work investigates the role of ion inertia in the presence of an axial magnetic field. It will be interesting to investigate the role of ion inertia on the mag-

netorotational instability for a more general field geometry, particularly in the context of profile independent destabilizing feature of ambipolar diffusion (Kunz & Balbus 2004). We shall leave this problem for future consideration.

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